Algebra II - BMath I
Mid Term Examination
Solutions

1. Decide whether the following statements are TRUE or FLASE. Answer that are not accompanied by correct justification will not be awarded any marks.

(a) There exists a 4×3 real matrix A and a 3×4 real matrix B such that the column vectors of AB are linearly independent.
Ans. False. Because a 4×3 matrix A can be treated as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ and hence its rank is ≤ 3. Similarly, B can be treated as a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ and hence its rank is ≤ 3. Therefore, $rank(AB) \leq 3$ as $rank(AB) \leq min\{rank(A), rank(B)\}$, and it cannot have all four columns linearly independent.

(b) If V is a vector space of dimension 8 and $T : V \longrightarrow V$ is a linear transformation such that $T \circ T = 0$, then $rank(T) \leq 4$.
Ans. True. We know that $rank(T) + nul(T) = 8$, where $nul(T)$ denote the dimension of the kernel of T. Let us write $T(V)$ to denote the image space of V under the action of T and we write $T|_{T(V)}$ to denote the restriction of T on the image space. Clearly $ker(T|_{T(V)}) \subseteq ker(T)$, hence $nul(T|_{T(V)}) \leq nul(T)$. Also, note that since $T \circ T = 0$, $ker(T|_{T(V)}) = T(V)$ and hence $nul(T|_{T(V)}) = rank(T)$. Therefore,

$$2\ rank(T) = rank(T) + nul(T|_{T(V)}) \leq rank(T) + nul(T) = 8,$$

hence $rank(T) \leq 4$.

(c) Suppose A and B are 2×2 real matrices such that $AX \neq BX$ for all non-zero X. Then $A - B$ is invertible.
Ans. True. If $A - B$ is not invertible then there will be a non-zero matrix C such that $(A - B)C = 0$. Since C is a non-zero matrix, there will be $v \in \mathbb{R}^2$, $v \neq 0$ such that $C \cdot v = w \neq 0$. Now,

$$A \cdot w - B \cdot w = (A - B) \cdot w = (A - B)C \cdot v = 0 \cdot v = 0,$$

a contradition to the given fact that $AX \neq BX$ for all non-zero X.

(d) Let $A = (a_{ij})$ be a square matrix with real entries such that $\sum_j a_{ij} = 1$, for all i. Then there exists a non-zero X such that $AX = X$.
Ans. True. Choose $X = (1, 1, \ldots, 1)^t$.

(e) If V is a vector space over a field F and $W \neq 0$ is a subspace such that $V/W \cong V$ then V is infinite dimensional.
Ans. True. For a finite dimensional vector space V, we will have $dim(V) = dim(V/W) = dim(V) - dim(W)$, where $dim(W) \neq 0$ as W is a non-zero subspace. This is not possible and hence V is infinite dimensional.

2(a). Show that the system $AX = B$, where $B = (1, 2, 3)^t$ and

$$\begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{pmatrix}$$

has no solution.
Ans. Reducing the rows we can see that the coefficient matrix A has rank 2. However, reducing rows we can see rank of the augmented matrix $(A|B)$ is 3. Therefore the system of above equations is inconsistent and will not have solutions.

2(b). Prove that a square matrix with entries from a field F is invertible if and only if its columns are linearly independent.
Ans. Let us consider a $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ . & \dots & . \\ . & \dots & . \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = (A_1, \dots, A_n)$$

where $A_1, \dots, A_n$ are n columns of A. An $n \times n$ matrix induces a linear transformation from F^n to F^n. First let us assume A is invertible. Therefore, A induces an isomorphism. If the columns of A are not linearly independent then we will find $b_1, \dots, b_n$, not all zero such that $b_1 A_1 + \dots + b_n A_n = 0$, i.e, $A(b_1, \dots, b_n)^t = (0, \dots, 0)^t$. Also, $A(0, \dots, 0)^t = (0, \dots, 0)^t$. This shows that the transformation induced by A is not injective. This is a contradiction and hence the columns of A are linearly independent.

Let us assume the columns of A are linearly independent and hence they form a basis for F^n. Let $e_1, \dots, e_n$ be the standard basis of F^n where e_i is a row vector of length n with 1 at the i^{th} position and zeros elsewhere. Since the columns of A form a basis for F^n, for each e_i there exists $b_{1i}, \dots, b_{ni}$ such that $A_1 b_{1i} + \dots + A_n b_{ni} = e_i$, i.e, $A(b_{1i}, \dots, b_{ni})^t = (e_i)^t$. Consider the matrix

$$B = \begin{pmatrix} b_{11} & \dots & b_{n1} \\ . & \dots & . \\ . & \dots & . \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

Note that $AB = I_n$, where I_n is the $n \times n$ identity matrix. Hence A is invertible.

3(a) Consider the vector space $V = \mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers. Exhibit explicitly a linearly independent set $L = (v_1, v_2, v_3)$ of vectors of V.
Ans. Note that $\mathbb{R}$ is an infinite dimensional vector space over $\mathbb{Q}$. Let us take $L = (\sqrt{2}, \sqrt{3}, \sqrt{5})$. We will show that this is a linearly independent set. Consider the equation $a\sqrt{2} + b\sqrt{3} + c\sqrt{5} = 0$, where $a, b, c \in \mathbb{Q}$. We will show that a, b, c are zeros. If any two of a, b, c are zero then the third has to be zero. If only one of a, b, c is zero then the other two have to be zero. For example if $a = 0$ then $b/c = -(\sqrt{5}/\sqrt{3})$, which is impossible as one side is rational and the other side is not. Now assume $a \neq 0, b \neq 0, c \neq 0$. Then $a\sqrt{2} + b\sqrt{3} = -c\sqrt{5}$. Squaring both sides we get $2a^2 + 3b^2 + 2ab\sqrt{6} = 5c^2$, which gives $\sqrt{6} = (5c^2 - 2a^2 - 3b^2)/2ab$, not possible. Hence, a, b, c are all zeros.

3(b) Let V be a vector space of all polynomials with real coefficients of degree at most 2. Let t be a fixed real number and define $p_1(x) = 1, p_2(x) = x + t, p_3(x) = (x + t)^2$. Prove that $B = (p_1, p_2, p_3)$ is a basis for V. If $p(x) = a_0 + a_1 x + a_2 x^2$, what are the coordinates of p in the ordered basis B?
Ans. Let us first show that B is linearly independent. Let $\alpha p_1 + \beta p_2 + \gamma p_3 = 0$, where $\alpha, \beta, \gamma \in \mathbb{R}$. Substituting p_1, p_2, p_3 we get

$$\alpha + \beta(x + t) + \gamma(x^2 + 2xt + t^2) = 0.$$

Equating coefficients of x^2, x and the constant term we get α, β, γ are zero and hence B is linearly independent. $p(x)$ is a generic element of V. To show that B spans V we will show that $p(x)$ can be expressed as a linear combination of elements of B with real coefficients. Let r, s, u be real nos. such that $rp_1 + sp_2 + up_3 = a_0 + a_1 x + a_2 x^2$, i.e,

$$r + s(x + t) + u(x^2 + 2xt + t^2) = a_0 + a_1 x + a_2 x^2.$$

Again equating coefficients we get $r = a_0 - a_1 t + a_2 t^2, s = a_1 - 2a_2 t, u = a_2$. Therefore B spans V and coordinates of $p(x)$ in the ordered basis B will be $(a_0 - a_1 t + a_2 t^2, a_1 - 2a_2 t, a_2)$.